Heterogeneous Agents in Macro Models

Solving Hetero-Models *With* Aggregate Uncertainty: Conceptual and Practical Issues

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Plan overall

Basic tools for solving heterogeneous agent models

1. motivation
2. refresher: solution methods for rep agent models
3. Aiyagari model: hetero-model without aggregate uncertainty
4. Krusell-Smith algorithm: introducing aggregate uncertainty
5. alternative solution methods: XPA, hybrid, S-S
6. continuous time: basic techniques

Plan for today

Conceptual and practical issues of heterogeneous agent models

- existence/uniqueness of equilibria?

Practical issues with associated solution methods

- choices to be made
- simulation methods
- accuracy tests

Intro into alternatives/extensions (if time permits)

- hybrid method, XPA, S-S

Some key issues

Equilibria

Very little theoretical results about equilibria in hetero-models

- existence, uniqueness, under which state variables?
- Miao (2006), Acikgoz (2016)
  - existence of recursive eq. under particular conditions

For numerical analysis less of an issue

- after solution, check accuracy!
- we assume a recursive equilibrium exists
  - or at least an accurate approximation to it

Working in continuous time makes more headway in this respect

- we'll talk about this on Friday

Multiplicity

More likely with heterogeneous agents

- my actions depend on what I think others will do
  - and they are different from me
- relatively easier to generate market externalities

Practical issues

Krusell-Smith algorithm

1. guess aggregate law of motion ($\psi_{\overline{\mu}}$)
   - implies values for K_t^D and thus r_t and w_t
2. solve individual problem with given aggregate law of motion
3. simulate economy and calculate moments of joint distribution
4. estimate $\hat{\psi}_{\overline{\mu}}$ implied by simulation
5. compare to previous guess
   - if $\hat{\psi}_{\overline{\mu}} = \psi_{\overline{\mu}} \rightarrow$ stop
   - if $\hat{\psi}_{\overline{\mu}} \neq \psi_{\overline{\mu}} \rightarrow$ update and go to 2

Choices to be made

There are several choices you must make

- how to solve individual problem?
- which moments to pick (and of what)?
- how to update the aggregate law of motion?
- how to simulate the economy?
- when to stop iterating?
- how to check for accuracy?
- non-trivial market clearing and imposing equilibrium?

Practical issues

Individual problem

How to solve individual problem

You can use your favorite solution method, but

- number of state variables has increased
  - standard individual state variables ($k_{i,t}$, $\epsilon_{i,t}$)
  - standard aggregate state variables (Z_t)
  - moments of the distribution you're tracking! (m_t)
    - the latter is there for forecasting K_t and prices
    - individual behavior depends on path of prices
    - $\rightarrow$ individual behavior depends on m_t!

How to solve individual problem

What trade-offs are you facing?

- projection methods
  - global solution
  - but curse of dimensionality
- perturbation
  - doesn't suffer from curse of dimensionality
  - only local solution
    - why could this be a worry here?

Practical issues

Aggregate law of motion

Which moments to pick?

Krusell-Smith used only mean of capital

- there is no rule about this → depends on application
- in their setup, means give sufficiently accurate results
  - if policy rules are *exactly linear* in levels
  - only mean necessary for computing next period's mean
    - distribution of wealth doesn't matter
  - in their setup, policy rules close to linear in levels

Which moments to pick?

Essentially trial and error

- use "bottom-up" approach
  - start with means
  - solve model (i.e. until aggregate law of motion converges)
  - check accuracy
  - if you're lucky, you're done
  - if not, adjust (number of) moments

Which moments to pick?

"Non-traditional" moments can be informative

- e.g. mass of agents around important cutoff

Past values of aggregate shocks

- easier to implement
- often quite informative about distribution
- intuition?

Aggregate law of motion for what?

Above we've used moments of joint distribution

- this implied path for K_t^D and in turn w_t and r_t

But it is really prices that agents care about

- specify aggregate law of motion directly for prices
- but, still need laws of motion for moments of interest!

How to update coefficients?

Let $\psi^{q}_{\overline{\mu}}$ be the coefficient guess in the qth iteration

- after solving and simulating the model economy
- obtain a time-series of the moments of interest m_t
- use this time-series to estimate coefficients
  - regress m_{t+1} on Z_t and m_t
  - obtain $\hat{\psi}_{\overline{\mu}}$
- update coefficient guess according to

$$\psi^{q+1}_{\overline{\mu}} = \lambda \hat{\psi}_{\overline{\mu}} + (1 - \lambda)\psi^{q}_{\overline{\mu}}$$

  - choosing λ is again tricky
  - strike compromise between speed and convergence

Practical issues

Simulation

Two ways how to simulate

Given individual policy rules

- simulate a large cross-section of agents
  - use Monte-Carlo integration to get moments
- use a grid method not requiring stochastic simulation
  - not introducing any sampling noise

Simulating a cross-section

- very intuitive and simple to implement
- however, quite computationally costly
  - need large cross-section
  - need long cross-section

Grid method

Policy rules $k_{i,t+1} = k(k_{i,t}, \epsilon_{i,t}, S_t)$ are given

- where S_t is aggregate state (including tracked moments)

Create fine grid of nodes $(\kappa_{i,j})$ for joint distribution

- $p_{i,j,t}$ is the beginning-of-period mass of agents with
  - $k_{i,t}$ capital holdings and
  - $\epsilon_{j,t}$ level of idiosyncratic productivity

Grid method

Assume an initial joint distribution

- first, focus on capital choice

1. for each node, figure out end-of-period capital choice

$$k_{i,j,t+1} = k(k_{i,t}, \epsilon_{j,t}, S_t)$$

2. assign beginning-of-period mass ($p_{i,j,t}$) to nodes
   - issue: no mass in between nodes
   - split mass proportionally between neighboring nodes

Grid method

Split beginning-of-period mass proportionally

$$\omega_{i,j,t} = \frac{k_{i,j,t+1} - \kappa_{i-1,j}}{\kappa_{i,j} - \kappa_{i-1,j}}$$

$$\overline{p}_{i-1,j,t} = \overline{p}_{i-1,j,t} + p_{i,j,t}(1 - \omega_{i,j,t})$$

$$\overline{p}_{i,j,t} = \overline{p}_{i,j,t} + p_{i,j,t}\omega_{i,j,t}$$

- $\overline{p}_{i,j,t}$ is end-of-period mass
- careful at end points
- careful, one grid point can "be filled" by many others!

Grid method

- second, focus on idiosyncratic productivity
  - use transition law to figure out
  - beginning-of-period joint distribution in next period

1. for each node, figure out next-periods mean productivity value

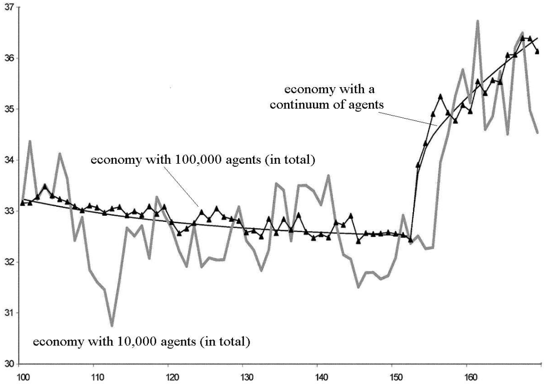
$$\epsilon_{j,t+1} = \rho\epsilon_{j,t}$$

2. assign mass end-of-period mass $(\bar{p}_{i,j,t})$ to nodes
   - according to distribution of idiosyncratic shocks, e.g. $N(0, \sigma_\epsilon^2)$
   - again split mass proportionally between nodes

Trade-offs with the above

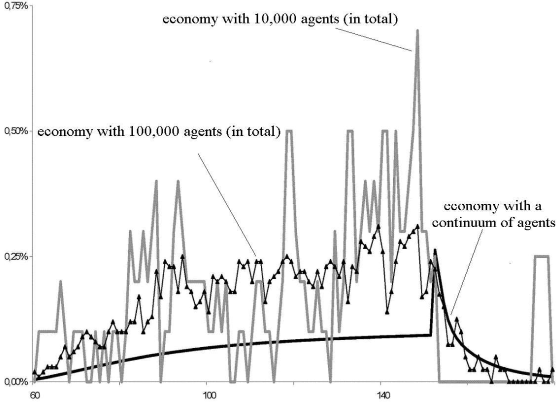
- Monte-Carlo simulation
  - can be computationally costly
  - introduces sampling noise
- Grid method
  - constructing the grid is not easy
    - may need many nodes
    - sometimes need precession in parts of state-space

When is simulation choice important (Algan et al., 2010)?



Notes: This graph plots the simulated aggregate capital stock of the unemployed using either a finite number (10,000) or a continuum of agents. It displays a subset of the observations shown in Figure 1.

When is simulation choice important (Algan et al., 2010)?



Notes: This graph plots the simulated fraction of unemployed agent at the borrowing constraint using either a finite number (10,000) or a continuum of agents.

Practical issues

Accuracy checks

When to stop?

When to stop what?

- when to stop iterating on aggregate law of motion?
- when to stop looking for accurate law of motion?
    - i.e. even if it has converged!

Updating procedure

Let $\psi_{\overline{\mu}}^{q}$ be the coefficient guess in the qth iteration

- after solving and simulating the model economy
- obtain a time-series of the moments of interest m_t
- use this time-series to estimate coefficients
  - regress m_{t+1} on Z_t and m_t
  - obtain $\hat{\psi}_{\overline{\mu}}$
- update coefficient guess according to

$$\psi_{\overline{\mu}}^{q+1} = \lambda \hat{\psi}_{\overline{\mu}} + (1 - \lambda) \psi_{\overline{\mu}}^{q}$$

When to stop updating?

Choice of λ

- strike compromise between speed and chance of convergence
- must specify a measure of (update) distance
  - e.g. max-abs-difference between updates

$$e^q = \max(\mathsf{abs}(\psi_{\mu}^{q} - \psi_{\mu}^{q-1}))$$

  - or often more conveniently in percentage terms

$$e^q = \max(\mathsf{abs}((\psi_{\mu}^{q} - \psi_{\mu}^{q-1})/\psi_{\mu}^{q-1}))$$

- stop when e^q falls below a certain threshold (e.g. 0.01%)

When to stop updating?

What does convergence of the aggregate law of motion mean?

- does not imply convergence to an accurate solution!
  - equilibrium prices actually depend on more moments
  - in a quantitatively important way
- $\rightarrow$ need to check for accuracy of entire solution

Accuracy checks

What dimensions of inaccuracy are there in our setup?

- numerical integration
  - we know that this is typically not an issue
- policy rules
  - depends on solution method
  - test for it by checking Euler equation errors
- aggregate law of motion
  - let's discuss that now

After we stop updating law of motion

We have

- individual policy rules
- initial joint distribution
- a simulation method
- a candidate aggregate law of motion

$$m_{t+1} = \overline{\mu}(Z_t, m_t; \psi_{\overline{\mu}}) + u_{t+1} \tag{1}$$

  - the above needs to be chekced for accuracy
  - true law of motion has $u_{t+1} = 0 \ \ \forall t$

Popular way to check for accuracy

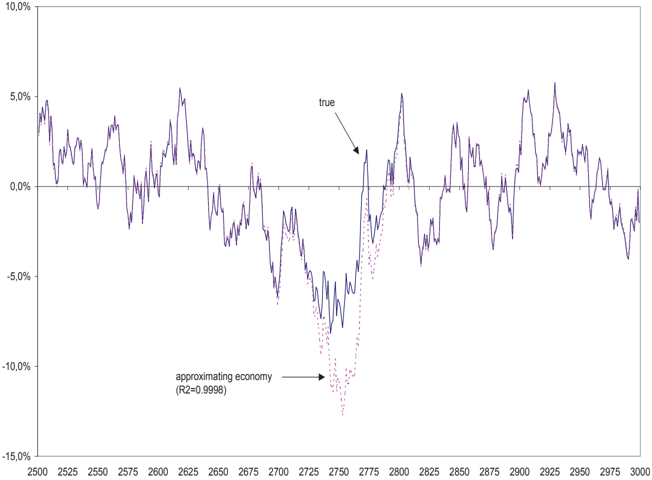
Use only individual policy rules

- simulate economy and obtain a time-series for m_t
- use these time-series to run regression (1)
- evaluate goodness of fit by looking at R^2
  - and sometimes standard error of regression

Issues with popular accuracy tests

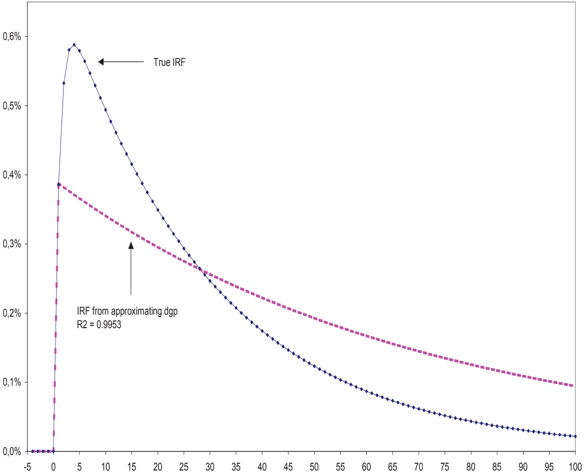
- R^2 and $\hat{\sigma}_u$ are averages
  - can mask potentially large, counteracting, differences
- not clear what is a low R^2
  - check out den Haan's work for fun examples!
- main problem is conceptual!
  - each period, the truth is used to update the approximation!
  - i.e. we're cutting the law of motion too much slack

Great R^2 but poor performance



Source: den Haan (2010)

Great R^2 but poor performance



Source: den Haan (2010)

Better accuracy tests (den Haan, 2010)

Generate two sequences of desired moments m_t

1. using only individual policy rules
2. using same aggregate shocks and only candidate law of motion

And compare the two sequences

- accuracy plot!
- compute $\max(\text{abs})$ difference and see where it occurs
  - important part of the state-space?
- what is the average/minimum error?
- look at IRFs under the two laws of motion etc.

Practical issues

Imposing equilibrium

Approaching equilibrium?

We've seen that the iterative algorithm approaches to (the) equilibrium

- is this always going to be the case?

Imagine the same economy, but with bonds in zero net supply

- how to solve for the aggregate bond price q_t?

Solving for the bond price

We could try to use an aggregate law of motion for q_t

1. guess coefficients of $q_t = q(S_t; \psi_q)$
2. solve individual problem
3. simulate economy
4. update coefficients ψ_q accordingly

So where's the problem?!

Imposing equilibrium

There is nothing ensuring zero net supply across iterations!

- departures from equilibrium are likely to accumulate!

Instead of q_t we can approximate something else to impose equilibrium

- $d(s_{i,t}) = b'(s_{i,t}) + q_t$
  - q_t is aggregate bond price
  - $b'(s_{i,t})$ is individual bond choice

How does this help?!

- $q_t = \sum_i d(s_{i,t})$ in *each* period
- i.e. are imposed to clear!

Summary

What did we do?

Key issues of heterogeneous agent models

- existence/uniqueness of equilibria?

Practical issues with associated solution methods

- choices to be made
- simulation methods
- accuracy tests

What's next?

An alternative solution methods/Extensions

- hybrid methods, XPA, S-S

Continuous time